

Wavefunctions for Non-Abelian Vortices

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Abstract

We construct exact wavefunctions of two vortices on a plane, a single vortex on the cylinder and a vortex on the torus. In each case, the physics is shown to be equivalent to a particle moving in a covering space, something simple to solve in those examples. We describe how our solutions fit into the general theory of quantum mechanics of N particles on a two-dimensional space and attribute our success to the fact that the fundamental groups are Abelian in those simple cases that we are considering.

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Vortices [1] are point-like topological defects on a two dimensional space. Consider the spontaneous symmetry breakdown of a simply-connected gauge group K into a finite subgroup G . This will give rise to stable topological vortices.¹ Neglecting any finite-core effects, the low energy theory is essentially a topological field theory with a finite gauge group G [2–5]. A vortex carries a “flux” which can be labeled by an element of G . If G is non-Abelian, vortices can exhibit topological interaction with one another: Adiabatically bringing a vortex around another will change the fluxes of both vortices even though they never come close to each other [6–8]. Even in the case of a single vortex, non-trivial topological interactions can still occur if the space is *not* simply-connected [9–11]. A multiply-connected space contains non-contractible loops. The flux of a vortex may change when it traverses those loops.

Both the vortex-vortex interactions and the interactions between a vortex and a non-contractible loop in space can be described by flat connections in configuration spaces. As is well-known, a flat connection can be trivialized at the expense of introducing multivalued wavefunctions (i.e., wavefunctions with non-trivial monodromy properties). This is the strategy we are adopting in this paper. We will find out those boundary conditions and construct exact wavefunctions for simple examples (two vortices on a plane and a single vortex on a cylinder or a torus). We also describe how our solutions fit into the general theory of the quantum mechanics of N particles on a two-dimensional space. Finally, we remark that our success in solving those simple cases is partly due to the fact that the fundamental groups of their configuration spaces are Abelian.

Consider the vortices that arise due to the symmetry breakdown of a gauge group into a finite group G . We assign an element of G to any isolated vortex to label its flux by the following method. Choose a fixed but *arbitrary* base point, x_0 , and a loop C that encloses the vortex and begins and ends at the point x_0 . Associate the vortex with the *untraced*

¹Vortices arise whenever the low energy gauge group of a symmetry broken theory has *disconnected* components. For simplicity, we take the low energy gauge group to be finite throughout this paper.

Wilson loop operator:

$$a(C, x_0) = P \exp \left(i \int_{C, x_0} A dx \right) , \quad (1)$$

where P denotes the path ordering. Since the gauge field is massive for a finite unbroken group, the element $a(C, x_0)$ is invariant under deformations of the path C that keep x_0 fixed and that avoid the vortex core. An object that transforms as an irreducible representation ν of G acquires an “Aharonov-Bohm” phase $D^\nu(a(C, x_0))$ when covariantly transported around the vortex. $a(C, x_0)$ has to be an element of G because the Higgs condensate must be invariant when so transported.

If there are two or more vortices, we must choose a standard loop for each vortex as in FIG. 1. Then we assign group elements a_1, a_2, \dots, a_n to the loops $\gamma_1, \gamma_2, \dots, \gamma_n$ respectively. This description is ambiguous because under a gauge transformation by $g \in G$ at the base point x_0 , the elements a_1, a_2, \dots, a_n transform according to $a_i \rightarrow g a_i g^{-1}$. For a single vortex, the gauge transformations act transitively on the conjugacy class of G to which a vortex belongs. Thus, one might be tempted to say that the flux of a vortex should really be labeled by a conjugacy class rather than a group element. But this is not correct because there is only one overall gauge degree of freedom. If there are two vortices, labeled by group elements a and b with respect to the same base point x_0 , then the effect of a gauge transformation at x_0 is $g : a \rightarrow g a g^{-1}, b \rightarrow g b g^{-1}$. Thus, if a and b are distinct representatives of the same class, they remain distinct in any gauge.

We consider only the non-relativistic quantum mechanics of non-interacting non-Abelian vortices. *Locally*, the Hamiltonian is just that of non-interacting particles

$$H = \sum_{i=1}^N \frac{p_i^2}{2m_i} . \quad (2)$$

The complexity of the problem, however, lies in the non-trivial monodromy properties that the wavefunctions must satisfy.

The case of a single vortex on an infinite plane is trivial because there is no topological interaction and the vortex behaves just like a free particle. A more interesting example

is two vortices on a plane. The center of mass motion can be separated from the relative motion. Passing to the center of mass frame, the relative motion is equivalent to that of a particle moving on the plane with the origin deleted.²

The wavefunctions for the relative motion have a non-trivial monodromy property. Standard paths γ_1 and γ_2 that wind counterclockwise around the two vortices have been chosen in FIG. 2a. Suppose now that vortex 1 winds around vortex 2 as in FIG. 2b. We may deform our paths during the winding so that no vortex ever crosses any path; then each path is mapped to the same group element after the winding as before the winding. But after the winding, the final deformed path is not homotopically equivalent to the initial path.

Suppose, for example, that initially γ_1 (γ_2 respectively) is mapped to a_1 (a_2 respectively). To determine the final values, after the winding, of the group elements associated with the paths γ_1 and γ_2 , notice that during the winding, the path shown in FIG. 2c is “dragged” into γ_1 . Hence, the group element associated with this path before the winding will become the element associated with γ_1 after the winding. Since this path is homotopically equivalent to $(\gamma_1\gamma_2)\gamma_1(\gamma_1\gamma_2)^{-1}$, where $\gamma_1\gamma_2$ denotes the path that is obtained by traversing γ_2 first and γ_1 second, before the winding, the path is associated with the element

$$a'_1 = (a_1a_2)a_1(a_1a_2)^{-1} . \quad (3)$$

Similarly, the path shown in FIG. 2d is dragged during the winding to γ_2 . This path is $(\gamma_1\gamma_2)\gamma_2(\gamma_1\gamma_2)^{-1}$ and, before the winding, it is mapped to

$$a'_2 = (a_1a_2)a_2(a_1a_2)^{-1} . \quad (4)$$

We conclude that, when a vortex of flux a winds counterclockwise around a vortex of flux b as shown in FIG. 2, the fluxes of both vortices will be conjugated by ab [6–8]: Denoting the fluxes of the two vortices before the winding by $|a, b\rangle$, on winding,

$$|a, b\rangle \rightarrow |(ab)a(ab)^{-1}, (ab)b(ab)^{-1}\rangle . \quad (5)$$

²We are imposing the hard-core condition that no two vortices can coincide.

If the two vortices wind around each other m times, from (5), their fluxes become

$$|(ab)^m a(ab)^{-m}, (ab)^m b(ab)^{-m}\rangle . \quad (6)$$

Since the unbroken group G is assumed to be finite, the fluxes eventually return to their original values, say after n windings, i.e., $(ab)^n a(ab)^{-n} = a$ and $(ab)^n b(ab)^{-n} = b$. Let us fix the a vortex at the origin. If the b vortex goes around the origin n times, the relative wavefunction will not change at all. Notice that in polar coordinates (r, θ) , the usual requirement of a periodicity of 2π for θ not longer applies. Owing to topological interactions, the required period of the relative wavefunction is $2\pi n$ rather than 2π [8]. (From the point of view of a vortex, the physical space is an n -sheeted surface [12] with an n that depends on a and b .)

One may still attempt to restrict θ to the range between 0 and 2π . If we denote the state of the fluxes of the two vortices after k windings by

$$|k\rangle = |(ab)^k a(ab)^{-k}, (ab)^k b(ab)^{-k}\rangle, \quad (7)$$

at each angle θ , the two vortices can be in one of the n flux eigenstates $|0\rangle, |1\rangle, \dots, |n-1\rangle$. Therefore, a wavefunction is represented by a column vector with n entries $\psi_0, \psi_1, \dots, \psi_{n-1}$ satisfying the relations

$$\psi_k(r, \theta + 2\pi) = \psi_{k+1}(r, \theta) . \quad (8)$$

It is convenient to transform to the “monodromy eigenstates.” A basis vector

$$\chi_l(r, \theta) = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} e^{-2\pi i k l / n} \psi_k(r, \theta) . \quad (9)$$

From Eq. (8), χ_l has the property that

$$\chi_l(r, \theta + 2\pi) = e^{2\pi i l / n} \chi_l(r, \theta) . \quad (10)$$

As discussed in Ref. [8], these monodromy eigenstates correspond to states of the two-vortex system that have definite charge, in the sense that they are eigenstates of the gauge

transformation $ab \in G$, where ab is the total flux.³

If two vortices are in the same conjugacy classes, the two vortices should be regarded as indistinguishable [8]. We should consider braiding \mathcal{R} (counterclockwise exchange) between the two instead of winding. It is possible that the braid operator has an orbit of odd order acting on the two vortex state. In that case, the wavefunction has a periodicity $(2n + 1)\pi$.

We now turn to the case of a single vortex on the cylinder. This case is similar to the previous one. Let us choose the arbitrary base point to be at “spatial infinity”. Choose a standard path γ around the vortex. There is a homotopy class of non-contractible loops on a cylinder. Call it α . Suppose the vortex winds along α as shown in FIG. 3a. The standard path γ will be dragged into a final deformed path $\gamma' = \alpha^{-1}\gamma\alpha$ in FIG. 3b. If initially α and γ are assigned with fluxes a and b , let us denote this assignment by $|a, b\rangle$. After the winding, the relation $\gamma' = \alpha^{-1}\gamma\alpha$ implies that the assignment changes to $|a, aba^{-1}\rangle$.

Again since G is finite, there is a minimal positive integer n such that $a^n b a^{-n} = b$. We can proceed in the same manner as in the case of two vortices on a plane and see that the quantum mechanics of a vortex on a cylinder of radius R is equivalent to that a free particle on a cylinder with a larger radius nR . A cylinder is diffeomorphic to an infinite plane with the origin deleted. Therefore, a vortex on a cylinder is topologically equivalent to two vortices on a plane. This is the reason why the physics is essentially the same in the two cases. One also notes that there are two separated infinite regions on the cylinder. Mapping either of the two to the origin of the plane gives equivalent results.

The last case that we will study is a vortex on the torus. Once again, let us choose an arbitrary base point. There are two homotopy classes of non-contractible loops on a torus. We denote them by α and β . There are magnetic fluxes associated with the two loops,

³The wavefunctions of two charge-flux composites (i.e., dyons) can similarly be obtained. The only difference is that, after n windings, the two-dyon state will return to itself only up to a phase, which will change the eigenvalues of the monodromy operator.

$\alpha \mapsto a$ and $\beta \mapsto b$. Let the flux of the vortex measured along the path γ be c . Let us denote the state of the fluxes of the vortex and the two non-contractible loops by $|c; a, b\rangle$. Since the space is compact, the fluxes a , b and c satisfy a relation [11]. In the convention of FIG 4a, the relation is

$$c = b^{-1}a^{-1}ba . \quad (11)$$

If the vortex goes around the α loop, it is easy to see in FIG. 4b that the paths have been smoothly deformed into

$$\alpha' = \alpha , \quad \beta' = \alpha^{-1}\beta\alpha , \quad \gamma' = \alpha^{-1}\gamma\alpha . \quad (12)$$

As the elements assigned to the deformed paths α' , β' and γ' will still be a , b and c , we find that the elements associated with the standard paths α , β and γ will be modified to

$$a' = a , \quad b' = aba^{-1} , \quad c' = aca^{-1} . \quad (13)$$

Notice that the effect of the winding is equivalent to a global gauge transformation by the element a . Because a torus is compact, if we regard our theory on the two-dimensional torus as fundamental, the Gauss law constraint for a compact surface demands that the state of the whole torus be invariant under global gauge transformations: A *closed* universe cannot carry any net gauge charges. An example of a state that satisfies the Gauss law constraint is, up to normalization,

$$\sum_{g \in G} |gcg^{-1}; gag^{-1}, gbg^{-1}\rangle . \quad (14)$$

The winding of the vortex around the α loop will, therefore, lead to no observable changes. A similar argument applies to the winding around the β loop. In conclusion, the quantum mechanics of a vortex on a torus is equivalent to that of a free particle on a torus of the same size.

Incidentally, our solution also resolves some complication concerning the base point. It was noted [7] that, in an n -vortex configuration, when a vortex winds around the base

point, the fluxes of all the vortices appear to be conjugated. However, if we regard the base point as arbitrary, winding around it should lead to no observable changes. To avoid this complication, it is convenient to place the base point at “spatial infinity”. However, for a compact surface like a torus, there is no “spatial infinity” to talk about. One can no longer ignore the possibility of the winding of a vortex around the base point on a torus. This will lead to a gauge transformation of the whole configuration. But there is an easy way out: the Gauss law for a compact surface precisely demands that all states related by gauge transformations to be identified. Hence, the base point is indeed arbitrary and winding around it leads to no observable changes.

We will now describe how our results for those special cases fit into the general theory of quantum mechanics of N particles on a two-dimensional space [13]. In general discussions of the quantum mechanics of N particles, the following framework is usually adopted: Suppose that the position of each particle takes value in a manifold M . If we allow no two particles to coincide (i.e., impose the hard-core condition), the classical configuration space for N *distinguishable* particles is $\mathcal{D}_N = M^N - \Delta$ where Δ is the subset of M^N in which at least two points in the Cartesian product coincide. For indistinguishable particles, we consider $\mathcal{C}_N = (M^N - \Delta)/S_N$ where S_N is the symmetric group of N elements. The configuration space, \mathcal{C}_N or \mathcal{D}_N , is typically not simply-connected. Suppose we quantize the theory by using the path integral formulation. The histories that contribute to the amplitude for a specified initial configuration to propagate to a specified final configuration divide up into disjoint sectors labeled by the elements of the fundamental group of the configuration space ($\pi_1(\mathcal{C}_N)$ or $\pi_1(\mathcal{D}_N)$). We have the freedom to weight the contributions from the different sectors with different factors, as long as the amplitudes respect the principle of conservation of probability. It can be shown that this requirement is equivalent to restricting the weighting factors to be unitary representations of $\pi_1(\mathcal{C}_N)$ or $\pi_1(\mathcal{D}_N)$.

Let us consider the special cases discussed earlier. A single vortex moving on a plane has a contractible configuration space and the quantum mechanics is, therefore, equivalent to that of a free particle.

For two distinguishable vortices on a plane, the fundamental group of the configuration space, $\pi_1(\mathcal{D}_2) = Z$, the set of integers. Notice that the monodromy eigenstates, χ_l , are irreducible representations of Z with eigenvalues, $e^{2\pi il/n}$. Similar arguments apply to the cases of two indistinguishable vortices or two dyons on a plane.⁴ The case of a vortex on a cylinder is topologically equivalent to two vortices on a plane.

The configuration space of a vortex on a torus is simply the torus itself, which has two non-contractible loops. Its fundamental group is, therefore, $Z \times Z$. Curiously, we see that only the trivial representation is realized in this case. This is true on a torus not just for a vortex, but also for a charged particle or a flux-charge composite (i.e., a dyon).⁵

We construct exact wavefunctions for two vortices on a plane and a vortex on a cylinder or a torus. Our success is partly due to the fact that the fundamental groups are Abelian in these cases. Any more complicated systems such as three vortices on a plane or two vortices on a cylinder or a torus involve configuration spaces with non-Abelian fundamental groups. In those cases, our simple arguments are doomed to fail. More powerful methods, yet to be developed, are needed for tackling those problems. Some of the results presented in this manuscript have also been obtained independently by Brekke, Collins and Imbo [14].

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⁴Incidentally, with our knowledge of the exact wavefunctions for two indistinguishable vortices, the second virial coefficient of non-Abelian vortices can be readily obtained.

⁵This conclusion is based on our assumption that our two-dimensional theory is fundamental and the Gauss law constraint is strictly satisfied. If, instead, we regard the two-dimensional theory as an effective theory of say a confined electron moving a two-dimensional thin film with periodic boundary conditions, there is no reason to impose such a strong assumption and the electron wavefunction may transform non-trivially under $Z \times Z$. (So long as the field strength vanishes, it remains true that the winding around the loop $\alpha\beta\alpha^{-1}\beta^{-1}$ leads to no observable changes.)

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FIGURES

FIG. 1. An arbitrary base point x_0 common to all vortices and a standard path γ_i , based at x_0 , around each vortex. The case of three vortices is shown in the figure.

FIG. 2. (a) Standard paths γ_1 and γ_2 , based at x_0 , that enclose vortex 1 and 2. (b) Vortex 1 winds around vortex 2. (c) Path that, during the winding of vortex 1 around vortex 2, gets dragged to γ_1 . (d) Path that gets dragged to γ_2 during the winding.

FIG. 3. (a) Path γ that encloses the vortex and path α that is non-contractible on a cylinder. The vortex winds around the cylinder. (b) Path γ gets dragged to γ' during the winding.

FIG. 4. (a) Paths α and β are non-contractible loops on a torus. Path γ that encloses the vortex. The vortex winds around the α loop. (b) Paths β and γ get dragged to β' and γ' , during the winding.